Lecture 15

Poisson Brackets

Defn: Poisson Brackets

$$\{F,G\} = \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$$

Note: $\{q_i, q_i\} = 0 = \{p_i, p_i\}$, $\{q_i, p_j\} = \delta_{ij}$

Note $[q] = L$ (position), $[p] = \frac{ML}{T}$ (momentum), $[qp] = \frac{ML^2}{T}$ energy x time (action)

$[\hbar] = \frac{ML^2}{T}$ Planck's constant

$\hat{q}\hat{p} - \hat{p}\hat{q}$ $\quad \hat{q}, \hat{p}$ operators

Dirac's quantization $\boxed{\frac{[q,p]}{i\hbar} \leftarrow \{p,h\}}$

(iv) Three functions on phase space F, G, H, then

$$\boxed{\{F,G,H\} = \{F,G\} + \{G,H\}}$$

(derivative properties)

(v) We also have

$$\boxed{\{F,G,H\} = \{F,G\}H + G\{F,H\}}$$

(vi) If F, G are both conserved, i.e. $\{F,H\} = 0$, $\{G,H\} = 0$ (Here H is hamiltonian)
then

$$\boxed{\{\{F,G\},H\} = 0}$$

(use Jacobi identity)

Note: Using Jacobi Identity, we have that

$$\{\{F,G\},H\} = -\{H,\{F,G\}\} = \{G,\underbrace{\{H,F\}}_{=0}\} + \{F,\underbrace{\{G,H\}}_{=0}\} = 0$$

(vii) Changing Co-ordinates on phase space

$$q_i \to Q_i(\underline{q},\underline{p}), \quad p_i \to P_i(\underline{q},\underline{p}) \qquad \text{(invertible)}$$

and require

$$\{Q_i, Q_j\} = 0 = \{P_i, P_j\}, \quad \{Q_i, P_j\} = \delta_{ij} \quad \to \text{canonical transformation}$$

to keep poisson structure

and also

$$\{F,G\}_{pq} = \{F,G\}_{PQ}$$

Note:
$$\frac{\partial}{\partial q_i} = \frac{\partial Q_j}{\partial q_i}\frac{\partial}{\partial Q_j} + \frac{\partial P_j}{\partial q_i}\frac{\partial}{\partial P_j}$$

$$\frac{\partial}{\partial p_i} = \frac{\partial Q_j}{\partial p_i}\frac{\partial}{\partial Q_j} + \frac{\partial P_j}{\partial p_i}\frac{\partial}{\partial P_j}$$

summation (Einstein's) Convention

Summed over $j = 1, \dots, N$

Then

$$\{F,G\}_{pq} = \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

$$= \frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial Q_\ell}\underbrace{\{Q_k, Q_\ell\}}_{=0} + \frac{\partial F}{\partial P_k}\frac{\partial G}{\partial P_\ell}\underbrace{\{P_k, P_\ell\}}_{=0}$$

$$+ \frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial P_\ell}\underbrace{\{Q_k, P_\ell\}}_{\delta_{k\ell}} + \frac{\partial F}{\partial P_k}\frac{\partial G}{\partial Q_\ell}\underbrace{\{P_k, Q_\ell\}}_{-\delta_{k\ell}}$$

$$= \frac{\partial F}{\partial Q_k}\frac{\partial G}{\partial P_k} - \frac{\partial F}{\partial P_k}\frac{\partial G}{\partial Q_k}$$

$$= \{F,G\}_{PQ}$$

Example 1: Put $Q_i = R_{ij} q_j$ (R_{ij} is independant of q, p)

summation convention for j

$$P_i = S_{ij} p_j$$

$$\{Q_k, Q_l\} = 0, \quad \{P_k, P_l\} = 0$$

$$\{Q_k, P_l\}_{pq} = \frac{\partial Q_k}{\partial q_i}\frac{\partial P_l}{\partial p_i} - \frac{\partial Q_k}{\partial p_i}\frac{\partial P_l}{\partial q_i}$$

$$= R_{ki} S_{li} - 0 \quad 0$$

$$= (RS^T)_{kl} = \delta_{kl}$$

$$\therefore S^T = R^{-1}$$

Example 2 (q, p) one degree of freedom

Put $\quad Q = q^\alpha \cos\beta p$

$\quad\quad P = q^\alpha \sin\beta p$

$$\{Q, Q\} = 0 = \{P, P\}$$

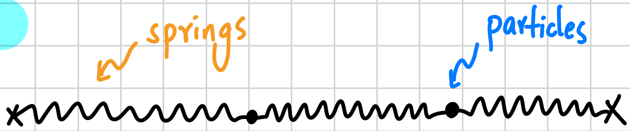
$$\{Q, P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}$$

$$= (\alpha q^{\alpha-1}\cos\beta p)(q^\alpha \beta\cos\beta p) - (-q^\alpha \beta \sin\beta p)(\alpha q^{\alpha-1}\sin\beta p)$$

$$= \alpha\beta q^{2\alpha-1}(\cos^2\beta p + \sin^2\beta p)$$

$$= \alpha\beta q^{2\alpha-1} = 1 = \begin{cases} \alpha = 1/2 \\ \beta = 2 \end{cases}$$

4.6 Normal Modes



Writing down the Hamiltonian

$$H(\underline{q},\underline{p}) = \frac{1}{2} p_i \underbrace{A_{ij}}_{\text{symmetric (eigenvalues} > 0)} p_j + \frac{1}{2} q_i \underbrace{B_{ij}}_{\text{symmetric (positive)}} q_j$$

(sum over i,j)

where A_{ij}, B_{ij} independenant of $\underline{q},\underline{p}$ and t

Since A is a symmetric matrix, we can diagonalize it. Put

$$A = R^T a R \ , \quad a \text{ is diagonal } \& \ RR^T = \mathbb{1}$$

So

$$\frac{1}{2} p_i A_{ij} p_j = \frac{1}{2} p_i (R^T a R)_{ij} \, p_j$$

and further, define

$$P_i = (\sqrt{a} R)_{ij} \, p_j \qquad Q_i = \left(\frac{1}{\sqrt{a}} R\right)_{ij} q_j \Rightarrow q_i = (R^T \sqrt{a})_{ij} \, Q_j$$

Then we get that

$$H = \frac{1}{2} P_i P_i + \frac{1}{2} Q_i \underbrace{(\sqrt{a} R B R^T \sqrt{a})}_{\text{symmetric}}{}_{ij} \, Q_j$$

Note: $(AB)^T = B^T A^T$

$R'^T \beta R'$ $\quad \beta$ is diagonal

Set $Q_i' = R_{ij}' \, Q_j$

$P_i' = R_{ij}' \, P_j$ Then we get that

$$H = \frac{1}{2} P_i' P_i' + \frac{1}{2} Q_i' \underbrace{\beta_{ij}}_{\text{diagonal}} Q_j' = \sum_{i=1}^{N} \left(\frac{1}{2} P_i' P_i' + \frac{1}{2} Q_i' Q_i' \beta_i \right)$$

Two transformations and at each stage chosen to be canonical

If $\beta_i > 0$ $i = 1, \dots, N$ then they determine frequencies of oscillations since Hamiltonian equations are

$$\dot{Q}_i' = \frac{\partial H}{\partial P_i'} = P_i' \qquad \dot{P}_i' = -\frac{\partial H}{\partial Q_i'} = \beta_i Q_i'$$

$$\Longrightarrow \ddot{Q}_i' = -\beta_i Q_i' \ , \ i \in \{1, \dots, N\}$$

$\longleftarrow$ solve for $Q_i'(t)$ $\rightarrow$ $q_i = (R^T \sqrt{a} R'^T)_{ij} \, Q_j'$

Normal Modes (from lecture notes)

Writing down the Hamiltonian, (summation convention)

$$H(\underline{q},\underline{p}) = \frac{1}{2} p_i \underset{\substack{\uparrow \\ \text{symmetric}}}{A_{ij}} p_j + \frac{1}{2} q_i \underset{\substack{\uparrow \\ \text{symmetric}}}{B_{ij}} q_j \qquad \text{(sum over } i,j\text{)}$$

(eigenvalues > 0) (positive)

A and B are real and symmetric $\Rightarrow$ eigenvalues are real and diagonalised by orthogonal transformations

If $\underline{q}$ and $\underline{p}$ are co-ordinates and momenta as components of a N-dimensional vector,

$$\boxed{H = \frac{1}{2}\underline{p}^T A \underline{p} + \frac{1}{2}\underline{q}^T B \underline{q}}$$

Since A is a symmetric matrix, we diagonalise it. Put

$$A = R^T a R \;, \quad a \text{ is diagonal } \& \quad RR^T = \mathbb{1}$$

Therefore we get

$$\frac{1}{2}\underline{p}^T A \underline{p} = \frac{1}{2}\underline{p}^T R^T a R \underline{p}$$

and further define

$$\underline{P} = \sqrt{a} R \underline{p} \;, \quad \underline{Q} = \frac{1}{\sqrt{a}} R \underline{q} \Rightarrow \underline{q} = R^T \sqrt{a}\, \underline{Q}$$

This is a canonical transformation and we get

$$H = \frac{1}{2}\underline{P}^T \underline{P} + \frac{1}{2}\underline{Q}^T \underbrace{\sqrt{a} R B R^T \sqrt{a}}_{\text{symmetric}} \underline{Q}$$

$R'^T \beta R'$, β diagonal $R'^T R' = \mathbb{1}$

Further define $\underline{Q}' = R' \underline{Q}$

$\underline{P}' = R' \underline{P}$ Then we get that

$$H = \frac{1}{2}\underline{P}'^T \underline{P}' + \frac{1}{2}\underline{Q}'^T \beta \underline{Q}' \equiv \frac{1}{2}\sum_{i=1}^{N}\left(P_i'^2 + \beta_i Q_i'^2\right)$$

The latter expression is the set of N independent harmonic oscillators

Two transformations and at each stage chosen to be canonical.

If $\beta_i > 0$, $i = 1, \dots, N$, they determine frequencies of oscillations since Hamiltonian equations are

$$\dot{Q}_i' = \frac{\partial H}{\partial P_i'} = P_i' \qquad \dot{P}_i' = -\frac{\partial H}{\partial Q_i'} = -\beta_i Q_i'$$

$$\Rightarrow \ddot{Q}_i' = -\beta_i Q_i', \; i \in \{1, \dots, N\} \longleftarrow \text{solve for } Q_i'(t)$$

$$q_i = (R^T \sqrt{a} R'^T)_{ij} Q_j'$$

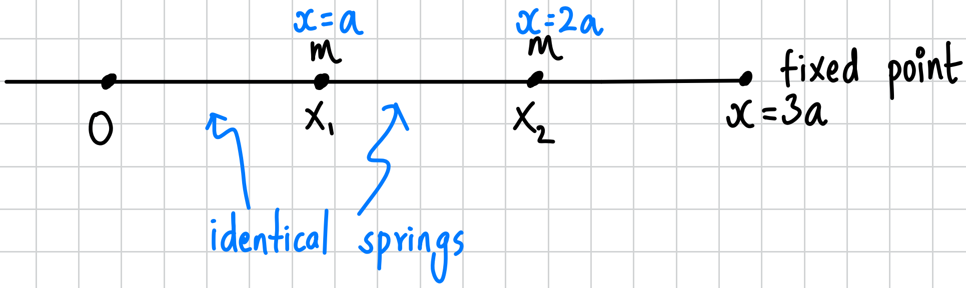
The components of diagonal matrix represent the set of possible frequencies of vibration for the system.

↳ These are called "normal modes" of the system.

All possible motions of the original system will be appropriate linear combination of these since

$$q = R^T \sqrt{a} R'^T Q'$$

Example Two particles, mass m, connected by springs



Set $x_1 = a + q_1$, $x_2 = 2a + q_2$, the Hamiltonian is

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k}{2}\left(q_1^2 + (q_2 - q_1) + q_2^2\right) \qquad k \text{ spring constant}$$

From normal modes section, we have

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{k}{2}\left(q_1^2 + (q_2 - q_1) + q_2^2\right) = \frac{1}{2m} p^T p + \frac{k}{2} q^T B q$$

$A = \mathbb{1}$

So we have

$$\begin{pmatrix} q_1 & q_2 \end{pmatrix} B \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = q_1^2 + (q_2 - q_1)^2 + q_2^2 \equiv \sum_{i,j=1}^{2} q_i B_{ij} q_j$$

$$\Rightarrow \boxed{B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}$$

To diagonalise this matrix, find eigenvalues

$$\det \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0 \Rightarrow 2 - \lambda = \pm 1$$

$$\Rightarrow \boxed{\lambda_1 = 1, \quad \lambda_2 = 3}$$

Finding eigenvectors,

$$\lambda_1 = 1 \qquad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \quad \Longrightarrow \quad u_1 = u_2 \Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 3 \qquad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \qquad u_1 = -u_2 \Rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Normalise eigenvectors:

$$\boxed{\underline{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \qquad \underline{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}}$$

Therefore define orthogonal matrix as

$$\boxed{R = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}} \Rightarrow R^T B R = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix} = \beta \qquad \left(\text{Here, } R = R^T\right)$$

Define canonical transformations

$$\underline{P} = R\underline{p} \quad , \quad \underline{Q} = R\underline{p} \qquad \left(A = \mathbb{1} \ , \ a = 1, \ \frac{1}{\sqrt{a}} = 1\right)$$

We get

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix}$$

$$\Rightarrow Q = \frac{1}{\sqrt{2}}\begin{pmatrix}q_1 + q_2\\q_1 - q_2\end{pmatrix} \Rightarrow \boxed{\begin{aligned} Q_1 &= \frac{1}{\sqrt{2}}(q_1 + q_2) \\ Q_2 &= \frac{1}{\sqrt{2}}(q_1 - q_2)\end{aligned}}$$

Similarly,

$$P_1 = \frac{1}{\sqrt{2}}(p_1 + p_2)$$

$$P_2 = \frac{1}{\sqrt{2}}(p_1 - p_2)$$

Writing hamiltonian in terms of P and Q

$$H = \frac{1}{2m}(P_1^2 + P_2^2) + \frac{k}{2}(Q_1^2 + 3Q_2^2) = \frac{1}{2m}\left(P_1^2 + P_2^2\right) + \frac{k}{2} Q^T \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix} Q^T$$

$$\Rightarrow H = \frac{1}{2m}(P_1^2 + P_2^2) + \frac{k}{2} Q^T \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix} Q \qquad \left(RBR^T = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}\right)$$

$$\Rightarrow H = \frac{1}{2m}(P_1^2 + P_2^2) + \frac{k}{2} Q^T \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix} Q$$

$$\Rightarrow H = \frac{1}{2}\sum_{i=1}^{2} P_i^2/m + K\beta_i Q_i^2$$

The hamiltonian equations are

$$\left.\begin{aligned} \dot{Q}_i &= \frac{\partial H}{\partial P_i} = \frac{1}{m}P_i \\ \dot{P}_i &= \frac{-\partial H}{\partial Q_i} = -K\beta_i Q_i \end{aligned}\right\} \Rightarrow \boxed{\ddot{Q}_i = -\frac{K}{m}\beta_i Q_i}$$

Frequencies are

$$\sqrt{\frac{K}{m}}, \sqrt{\frac{3K}{m}}$$

If we had N particles,

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 & 1\ 2 & -1 \\ \cdots & \cdots & \cdots & \cdots & -1 & 2 \end{pmatrix}$$

Eigenvalues

$$\lambda_k = 4\sin^2\frac{k\pi}{2(N+1)} \qquad k \in \{1, \dots, N\}$$

Note Let A be a matrix, $\underline{x}$ a vector

$$(A\underline{x})_i = A_{ij}x_j \Rightarrow A\underline{x} = \sum_{i,j=1}^{n} A_{ij}x_j$$

$$\underline{x}^T A\underline{x} = \sum_{i,j=1}^{n} x_i A_{ij} x_j$$

For eigenvalues and eigen vectors $A\underline{x} = \lambda\underline{x}$

$$(A\underline{x})_i = (\lambda\underline{x})_i \Rightarrow A_{ij}x_j = \lambda x_i$$

$$\Rightarrow \sum_{j=1}^{n} A_{ij}x_j = \lambda x_i$$